Applied Mathematics Letters 26 (2013) 296–300

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On some Ostrowski-like type inequalities involving n knots☆

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ARTICLE INFO

Article history:
Received 16 August 2012
Accepted 19 September 2012

Keywords:
Ostrowski inequality
Numerical integration
Error bound

ABSTRACT

In this paper, we study some Ostrowski-like type inequalities proposed by Huy and Ngô. We give some improvements of these inequalities which are inspired by a result established recently.

1. Introduction

The estimate of error for numerical integration is an important issue for many applications. Classical quadrature rules and their error estimates have been studied extensively [1–6]. A new type of quadrature rule was established recently by Huy and Ngô [7,8]. For $1 < m, n < \infty$, the quadrature in [7,8] involved knots $x_i \in (0, 1),\ i = 1, \ldots, n$ which are chosen to satisfy

$$\begin{aligned} &x_1 + x_2 + \cdots + x_n = \frac{n}{2}, \\ &\vdots \\ &x_1^m + x_2^m + \cdots + x_n^m = \frac{n}{m+1}. \end{aligned} \tag{1}$$

The error between the quadrature $Q(f, n, m, x_1, \ldots, x_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(a + x_i(b-a))$ and the integral $I(f) = \int_a^b f(x)dx$ of a function f was estimated by an Ostrowski-like inequality. Namely the following was established in [8].

Theorem 1.1. *Suppose that $f : [a, b] \to R$ is an m-th differentiable function. If $x_1, \ldots, x_n$ satisfy (1) for $j = 1, \ldots, m$, then we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \le \frac{2m+5}{4} \frac{(b-a)^{m+1}}{(m+1)!}(S - s), \tag{2}$$

where $S = \sup_{a \le x \le b} f^{(m)}(x)$, $s = \inf_{a \le x \le b} f^{(m)}(x)$.

☆ This research is supported by the grant MYRG086(Y1-L2)-FST12-VSW from University of Macau.

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doi:10.1016/j.aml.2012.09.012

The bound was later improved to be $\frac{(b-a)^{m+1}}{(m+1)!}(S-s)$ by the second author of this note [9]. Very recently, this Ostrowski-like inequality was further studied [10]. By considering a change of variable $y_i = x_i - \frac{1}{2}$, Xiao proved that $|I(f) - Q(f, n, m, x_1, \ldots, x_n)|$ is bounded by $\frac{1}{2^m}\frac{(b-a)^{m+1}}{(m+1)!}(S-s)$ when m is even.

Inspired by the work of Xiao, in this note, we give further studies of this Ostrowski-like inequality. In Section 2, we give an improved bound of (2) when m is odd and in Section 3, we give an improvement of another Ostrowski-like inequality introduced in [7].

2. Improvement for odd order

In this section, we assume that $m > 1$ is an odd number. We have the following.

Theorem 2.1. *Under the conditions of Theorem 1.1, if m is odd, then*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \le \left(1 + \frac{m+1}{m}\right)\frac{(b-a)^{m+1}}{(m+1)!2^{m+1}}(S-s).$$

Proof. In [10], it was proved that $y_i = x_i - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2}),\ i = 1, \ldots, n$, satisfy

$$\begin{aligned} &y_1 + y_2 + \cdots + y_n = 0,\\ &\vdots\\ &y_1^{m-1} + y_2^{m-1} + \cdots + y_n^{m-1} = \frac{n}{m}\frac{1+(-1)^{m-1}}{2^m},\\ &y_1^m + y_2^m + \cdots + y_n^m = \frac{n}{m+1}\frac{1+(-1)^m}{2^{m+1}}. \end{aligned} \tag{3}$$

Since $m > 1$, there must be at least one y_j which is nonzero. It is no harm to assume that, $y_1 \le y_2 \le \cdots y_\ell \le 0 \le y_{\ell+1} \le \cdots \le y_n$ for an ℓ satisfying $1 < \ell < n$.

Without loss of generality, we assume that

$$y_1^{m-1} + \cdots + y_\ell^{m-1} = \min\{y_1^{m-1} + \cdots + y_\ell^{m-1}, y_{\ell+1}^{m-1} + \cdots + y_n^{m-1}\} \le \frac{n}{2m}\frac{1+(-1)^{m-1}}{2^m} = \frac{n}{m2^m}.$$

We can thus conclude that $|y_1^m + y_2^m + \cdots + y_\ell^m| \le |y_1|y_1^{m-1} + \cdots + |y_\ell|y_\ell^{m-1} \le \frac{n}{m2^{m+1}}$. From the last equation of (3), we immediately get

$$y_{\ell+1}^m + \cdots + y_n^m \le \frac{n}{m2^{m+1}}.$$

Denoting $Q = Q(f, n, m, x_1, \ldots, x_n)$, the following equality was proved in [10]:

$$\begin{aligned} I(f) - Q = {} & \int_0^{\frac{b-a}{2}} \frac{\left(\frac{b-a}{2} - t\right)^m f^{(m)}\left(\frac{a+b}{2} + t\right)}{m!}dt + \int_{\frac{a-b}{2}}^0 \frac{\left(\frac{a-b}{2} - t\right)^m f^{(m)}\left(\frac{a+b}{2} + t\right)}{m!}dt\\ & - \frac{b-a}{n}\sum_{i=1}^n {y_i}^m \int_0^{b-a} \frac{(b-a-u)^{m-1}}{(m-1)!} f^{(m)}\left(\frac{a+b}{2} + y_i u\right)du. \end{aligned} \tag{4}$$

One can easily see that the following bounds hold:

$$\begin{aligned} & s\frac{(b-a)^{m+1}}{(m+1)!2^{m+1}} \le \int_0^{\frac{b-a}{2}} \frac{\left(\frac{b-a}{2} - t\right)^m f^{(m)}\left(\frac{a+b}{2} + t\right)}{m!}dt \le S\frac{(b-a)^{m+1}}{(m+1)!2^{m+1}},\\ & -S\frac{(b-a)^{m+1}}{(m+1)!2^{m+1}} \le \int_{\frac{a-b}{2}}^0 \frac{\left(\frac{a-b}{2} - t\right)^m f^{(m)}\left(\frac{a+b}{2} + t\right)}{m!}dt \le -s\frac{(b-a)^{m+1}}{(m+1)!2^{m+1}},\\ & S\frac{(b-a)^m}{m!}\sum_{i=1}^{\ell} {y_i}^m \le \sum_{i=1}^{\ell} {y_i}^m \int_0^{b-a} \frac{(b-a-u)^{m-1}}{(m-1)!} f^{(m)}\left(\frac{a+b}{2} + y_i u\right)du \le s\frac{(b-a)^m}{m!}\sum_{i=1}^{\ell} {y_i}^m,\\ & s\frac{(b-a)^m}{m!}\sum_{i=\ell+1}^{n} {y_i}^m \le \sum_{i=\ell+1}^{n} {y_i}^m \int_0^{b-a} \frac{(b-a-u)^{m-1}}{(m-1)!} f^{(m)}\left(\frac{a+b}{2} + y_i u\right)du \le S\frac{(b-a)^m}{m!}\sum_{i=\ell+1}^{n} {y_i}^m. \end{aligned}$$

Therefore

$$I(f)-Q \le \frac{(b-a)^{m+1}}{(m+1)!2^{m+1}}(S-s) - S\frac{(b-a)^{m+1}}{nm!}\sum_{i=1}^{\ell} {y_i}^m - s\frac{(b-a)^{m+1}}{nm!}\sum_{i=\ell+1}^{n} {y_i}^m$$
$$= \frac{(b-a)^{m+1}}{(m+1)!2^{m+1}}(S-s) + (S-s)\frac{(b-a)^{m+1}}{nm!}\sum_{i=\ell+1}^{n} {y_i}^m.$$

Similarly, one has $I(f)-Q \ge \frac{(b-a)^{m+1}}{(m+1)!2^{m+1}}(s-S)+(s-S)\frac{(b-a)^{m+1}}{nm!}\sum_{i=\ell+1}^{n} {y_i}^m$. Combining these two inequalities, the theorem follows. □

3. Improvement of another Ostrowski-like inequality

In this section, we study another Ostrowski-like inequality introduced in [7]. This inequality has been studied in [9] and we find that it can be further improved by the idea of Xiao in [10]. Namely, we get the following:

Theorem 3.1. *Suppose that $f : [a, b] \to R$ is an m-th differentiable function such that $f^{(m)} \in L^p(a, b)$. If $x_1, \ldots, x_n$ satisfy (1) for $j = 1, \ldots, m$, then we have*

$$|I(f)-Q| \le \begin{cases} \left(\frac{1}{2}\right)^m \frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + 2^{\frac{1}{p}}\left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}, & \text{when } m \text{ is odd} \\ \left(\frac{1}{2}\right)^m \frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + 2^{\frac{1}{p}}\left(\frac{m}{m-1}\right)^{\frac{1}{p}}\left(\frac{m}{(m+1)[(m-1)q+1]}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}, & \text{when } m \text{ is even} \end{cases}$$

for $p \in (1, \infty]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The theorem can be obtained by estimating the absolute values of the terms in (4).

Let $x = t + \frac{b-a}{2}$, we get $\int_{\frac{a-b}{2}}^{0} \frac{|\frac{a-b}{2}-t|^m |f^{(m)}\left(\frac{a+b}{2}+t\right)|}{m!}dt = \int_0^{\frac{b-a}{2}} \frac{x^m|f^{(m)}(a+x)|}{m!}dx$.

By the Hölder inequality, it follows that

$$\int_0^{\frac{b-a}{2}} \frac{t^m|f^{(m)}(a+t)|}{m!}dt \le \frac{1}{m!}\left(\int_0^{\frac{b-a}{2}} t^{mq}dt\right)^{\frac{1}{q}}\left[\int_0^{\frac{b-a}{2}} |f^{(m)}(a+t)|^p dt\right]^{\frac{1}{p}}$$
$$= \frac{1}{m!}\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^{m+\frac{1}{q}}\left[\int_a^{\frac{a+b}{2}} |f^{(m)}(t)|^p dt\right]^{\frac{1}{p}} \tag{5}$$

and

$$\int_0^{\frac{b-a}{2}} \frac{\left(\frac{b-a}{2}-t\right)^m \left|f^{(m)}\left(\frac{a+b}{2}+t\right)\right|}{m!}dt \le \frac{1}{m!}\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^{m+\frac{1}{q}}\left[\int_{\frac{a+b}{2}}^{b} |f^{(m)}(t)|^p dt\right]^{\frac{1}{p}}. \tag{6}$$

In order to estimate the terms on the right of (5) and (6), consider $g(x) = x^{\frac{1}{p}} + (1-x)^{\frac{1}{p}}$. One can easily check that, on the interval $(0, 1)$, the maximum of g is $g_{\max} = g\left(\frac{1}{2}\right) = 2^{1-\frac{1}{p}} = 2^{\frac{1}{q}}$. Since $\alpha = \frac{1}{\|f^{(m)}\|_p^p}\int_a^{\frac{a+b}{2}} |f^{(m)}(t)|^p dt \in (0, 1)$, the inequality $g(\alpha) \le g_{\max}$ implies

$$\left[\int_a^{\frac{a+b}{2}} |f^{(m)}(t)|^p dt\right]^{\frac{1}{p}} + \left[\int_{\frac{a+b}{2}}^{b} |f^{(m)}(t)|^p dt\right]^{\frac{1}{p}} \le 2^{\frac{1}{q}}\|f^{(m)}\|_p.$$

Combining this with the sum of (5) and (6), we have

$$\int_0^{\frac{b-a}{2}} \frac{\left(\frac{b-a}{2}-t\right)^m \left|f^{(m)}\left(\frac{a+b}{2}+t\right)\right|}{m!}dt + \int_{\frac{a-b}{2}}^{0} \frac{\left|\frac{a-b}{2}-t\right|^m \left|f^{(m)}\left(\frac{a+b}{2}+t\right)\right|}{m!}dt$$
$$\le \frac{1}{m!}\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^{m+\frac{1}{q}} 2^{\frac{1}{q}}\|f^{(m)}\|_p = \left(\frac{1}{2}\right)^m \frac{1}{m!}\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}.$$

Now we turn to the third term in (4), by the Hölder inequality again, we have

$$\begin{aligned}
&\frac{b-a}{n}\sum_{i=1}^{n}|y_i|^m\int_0^{b-a}\frac{(b-a-u)^{m-1}}{(m-1)!}\left|f^{(m)}\left(\frac{a+b}{2}+y_iu\right)\right|du\\
&\le\frac{b-a}{n}\frac{1}{(m-1)!}\sum_{i=1}^{n}|y_i|^m\left[\int_0^{b-a}(b-a-u)^{(m-1)q}du\right]^{\frac{1}{q}}\left[\int_0^{b-a}\left|f^{(m)}\left(\frac{a+b}{2}+y_iu\right)\right|^p du\right]^{\frac{1}{p}}\\
&=\frac{b-a}{n}\frac{1}{(m-1)!}\frac{(b-a)^{m-1+\frac{1}{q}}}{[(m-1)q+1]^{\frac{1}{q}}}\sum_{i=1}^{n}|y_i|^{m-\frac{1}{p}}\left|\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+y_i(b-a)}|f^{(m)}(t)|^p dt\right|^{\frac{1}{p}}\\
&\le\frac{1}{(m-1)!}\left[\frac{1}{(m-1)q+1}\right]^{\frac{1}{q}}\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}\sum_{i=1}^{n}\frac{|y_i|^{m-\frac{1}{p}}}{n}.
\end{aligned}$$

Notice that

$$\sum_{i=1}^{n}\frac{|y_i|^{m-\frac{1}{p}}}{n}=\sum_{i=1}^{n}\frac{|y_i|^{m-1+\frac{1}{q}}}{n}=\sum_{i=1}^{n}\frac{|y_i|^{\frac{m-1}{p}}}{n^{\frac{1}{p}}}\frac{|y_i|^{\frac{m-1+1}{q}}}{n^{\frac{1}{q}}}\le\left(\sum_{i=1}^{n}\frac{|y_i|^{m-1}}{n}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\frac{|y_i|^{m}}{n}\right)^{\frac{1}{q}}.$$

When m is odd, by the fact $|y_i|\le\frac{1}{2}$, we get

$$\left(\sum_{i=1}^{n}\frac{|y_i|^{m-1}}{n}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\frac{|y_i|^{m}}{n}\right)^{\frac{1}{q}}\le\left(\frac{1}{2}\right)^{\frac{1}{q}}\sum_{i=1}^{n}\frac{|y_i|^{m-1}}{n}=\frac{1}{m}\left(\frac{1}{2}\right)^{m-\frac{1}{p}},$$

which implies the first half of the theorem. On the other hand, when m is even, we get

$$\begin{aligned}
\left(\sum_{i=1}^{n}\frac{|y_i|^{m-1}}{n}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\frac{|y_i|^{m}}{n}\right)^{\frac{1}{q}}&\le\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\frac{|y_i|^{m-2}}{n}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\frac{|y_i|^{m}}{n}\right)^{\frac{1}{q}}\\
&\le\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{m-1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{m-2}}\right)^{\frac{1}{p}}\left(\frac{1}{m+1}\right)^{\frac{1}{q}}\left(\frac{1}{2^m}\right)^{\frac{1}{q}}\\
&=\frac{1}{m}\left(\frac{m}{m-1}\right)^{\frac{1}{p}}\left(\frac{m}{m+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{m-\frac{1}{p}}.
\end{aligned}$$

This concludes the second half of the theorem. □

Remark. The estimate given in Theorem 3.1 improves that in [9]. In fact, in [9], the error is shown to be bounded by

$$B(f,m,p)=\frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}+\left(\frac{m}{(m+1)[(m-1)q+1]}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}.$$

However, when m is odd, we have

$$\begin{aligned}
&\left(\frac{1}{2}\right)^m\frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}+2^{\frac{1}{p}}\left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}\\
&<\frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{m-\frac{1}{p}}\left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}\\
&\le\frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}+\left(\frac{m}{m+1}\right)^{m-\frac{1}{p}}\left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}\\
&\le\frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}+\left(\frac{m}{m+1}\right)^{1-\frac{1}{p}}\left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}=B(f,m,p).
\end{aligned}$$

At the same time, when m is even, the error bound in Theorem 3.1 satisfies

$$\left(\frac{1}{2}\right)^m \frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + 2^{\frac{1}{p}}\left(\frac{m}{m-1}\right)^{\frac{1}{p}}\left(\frac{m}{(m+1)[(m-1)q+1]}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}$$
$$< \frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + \left(\frac{1}{2}\right)^{m-\frac{1}{p}}\left(\frac{m}{m-1}\right)^{\frac{1}{p}}\left(\frac{m}{(m+1)[(m-1)q+1]}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}$$
$$\le \frac{1}{m!}\left[\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + \left(\frac{1}{2}\right)^{m-\frac{2}{p}}\left(\frac{m}{(m+1)[(m-1)q+1]}\right)^{\frac{1}{q}}\right]\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}},$$

which is also not greater than $B(f, m, p)$.

We conclude our discussion with an example demonstrating the results obtained in this note.

Example. In [6], it was shown that if $f^{(m-1)}$ is absolutely continuous and $s_m \le f^{(m)} \le S_m$ a.e. on $[a, b]$ for $m = 1, 2, 3$, then the following Simpson–Grüss type inequalities hold:

$$\left|I(f) - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \le C_m(S_m - s_m)(b-a)^{m+1},$$

where

$$C_1 = \frac{5}{72}, \qquad C_2 = \frac{1}{162}, \qquad C_3 = \frac{1}{1152}.$$

These constants are sharp in the sense that one cannot replace them with smaller numbers. The coefficients given by the estimate of Theorem 2.1 are $C_1' = \frac{3}{8}$, $C_3' = \frac{7}{1152}$. One can easily check that Theorem 2.1 gives closer values to the sharp ones than those given in [9].

Next suppose that f is 2 times differentiable. Theorem 3.1 gives

$$\left|I(f) - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \le \frac{1}{12}\|f''\|_\infty(b-a)^3.$$

If, in addition, f is 3 times differentiable, the theorem yields

$$\left|I(f) - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \le \frac{7}{576}\|f'''\|_\infty(b-a)^4.$$

Notice that, by the result in [9], one can get similar estimates but the constants are replaced by $\frac{1}{3}$ and $\frac{1}{12}$ for $m = 2$ and 3 respectively.

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